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Time ordered perturbation theory for non-local interactions; applications to NCQFT

Stefan Denk* **and Manfred Schweda**†

Institute for Theoretical Physics, Vienna University of Technology
Wiedner Hauptstrasse 8–10, A-1040 Vienna, Austria
E-mail: denk@hep.itp.tuwien.ac.at, mschweda@tph.tuwien.ac.at

Abstract: In the past decades, time ordered perturbation theory was very successful in describing relativistic scattering processes. It was developed for local quantum field theories. However, there are field theories which are governed by non-local interactions, for example non-commutative quantum field theory (NCQFT). Filk [1] first studied NCQFT perturbatively obtaining the usual Feynman propagator and additional phase factors as the basic elements of perturbation theory. However, this treatment is only applicable for cases, where the deformation of space-time does not involve time. Thus, we generalize Filk's approach in two ways: First, we study non-local interactions of a very general type able to embed NCQFT. And second, we also include the case, where non-locality involves time. A few applications of the obtained formalism will also be discussed.

Keywords: Renormalization Regularization and Renormalons, Non-Commutative Geometry.

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Contents

1. Introduction

Quantum field theory (QFT) on non-commutative space-time shows completely unexpected and fascinating features in the perturbative realization of the various non-commutative field theories like the ordinary scalar field theory invoked as a toy model, the gauge-field models (with and without supersymmetry) etc. Especially, the non-planar part of the tadpole graph of a scalar NCQFT in 4-dimensional space-time shows that the non-commutative phases of the four point interaction create an UV-finite result connected with a new type of IR-singularity for vanishing external momenta (naive power counting arguments would lead to the conclusion that these contributions are quadratically divergent — independent of the external momenta). This is the so-called UV/IR mixing problem [2]. Similar effects are also known in gauge field models [3]–[5].

A further alternative for describing non-commutative Yang-Mills theory is given with the Seiberg-Witten map [6]. It allows to connect the non-commutative gauge field with the ordinary gauge field. Both fields are unified with the "gauge-equivalent" gauge transformation.

In order to carry out perturbative calculations for NCQFT, one can use a set of modified Feynman rules first formulated in [1] and elaborated in great detail in [7] for the one-loop and two-loop approximation of a scalar field model. However, at this point, one has to stress that the corresponding deformation parameter $\theta_{\mu\nu}$ characterizing the non-commutativity of space-time must be restricted to the case $\theta_{0i} = 0$. This excludes the non-locality in the time direction. The only place, where one encounters time derivatives is the kinetic term of the action and the usual formalism to get Feynman rules is applicable. The case $\theta_{0i} \neq 0$ is more delicate due to the fact that non-localities are also present in the time direction. This is really an obstacle for the naive application of the perturbation theory à la Feynman, where simply the Feynman propagators of the quantum field models are associated with the Wick contractions. As will be seen below, this is not the case when time is also involved into non-locality. Motivated by the works [8]–[10], our investigations are devoted to the discussion of the more general case where non-locality also occurs in the time direction including NCQFT with a non-vanishing θ_{0i}. In [10], a compatible time ordering called *Interaction Point Time Ordered Perturbation Theory (IPTOPT)* was proposed in order to explicitely evaluate the Gell-Mann-Low formula proposed in [11]. The corresponding results found in [8]–[10] are very different from those in the naive approach treated in the literature. It is also claimed that with the help of IPTOPT, no unitarity problems arise. As it is shown in [12], the violation of unitarity is due to an improper definition of QFT on non-commutative space-time. Additionally, [13] also deals with the unitarity problem at the functional level.

A further advantage of the proposed IPTOPT is possibly the lack of the UV/IR mixing problem [10, 14]. In addition, one also has to mention that non-local interactions become important in the discussion of ultra-violet finite QFT [15]. Therefore, we investigate the perturbation theory of a very general, non-local interaction of scalar fields allowing the embedding of NCQFT, UV finite QFT, etc. The main aim of the presented paper is devoted to find "simply" modified Feynman rules for IPTOPT in the sense of [1, 7], which are also applicable for cases where non-locality involves time (e.g. $\theta_{0i} \neq 0$). With the help of these new, modified Feynman rules, it will be demonstrated for some special cases that the results [8, 10] can be reproduced very shortly and compactly in a complete new manner.

The paper is organized as follows: in section 2, we present the main ideas of a non-local, scalar QFT. Section 3 deals with new Feynman rules for general non-local interactions. With the help of a simple example, the "contractor" will be introduced in analogy to the ordinary covariant Feynman propagator of local QFT. In section 3.1, the Feynman rules are presented in coordinate space. The explicit expression for the contractor is elaborated in section 3.2. It can be seen immediately that one receives the well known Feynman propagator in a certain limit. Energy-momentum conservation at each vertex is discussed in 3.3. The corresponding Feynman-rules in momentum space are given in 3.4. In section 4.1, some specific examples of NCQFT are studied. The calculation of the tadpole very powerfully shows the efficiency of the new Feynman rules in comparison to the older methods presented in [10]. At the end of section 4, we apply our approach to the non-local interaction of UV-finite QFT [15]. A short discussion is presented in section 5. In appendix A, one finds the proof of the generalized Wick theorem for non-locally time-ordered vacuum expectation values.

2. Non-local interactions of general type

The starting point of NCQFT is based on

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,, \tag{2.1}$$

where the hermitean space-time operators are denoted by $\hat{x}^\mu$. The antisymmetric deformation parameter $\theta_{\mu\nu}$ is assumed constant here and has dimension $[\text{length}]^2$. In order to deal with the algebra eq. (2.1), one introduces the so-called (Weyl-Moyal) star product defined by [1, 7]

$$(f * g)(x) \equiv e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^\zeta \partial_\nu^\eta} f(x+\zeta)g(x+\eta)\Big|_{\zeta=\eta=0}\,. \tag{2.2}$$

In eq. (2.2), x^μ is an element of the usual commutative space-time and $f(x)$, $g(x)$ stand for the usual operators of commutative theory like the scalar field $\phi(x)$ for example. The self interaction density of a ϕ^4-theory would be modified the following way:

$$\mathcal{H}(z) = \frac{\kappa}{4!}\phi(z)^4 \to \frac{\kappa}{4!}\phi(z) * \phi(z) * \phi(z) * \phi(z)\,. \tag{2.3}$$

This is related to the interaction $V(z^0)$ in the interaction picture by $V(z^0) = \int d^3z \mathcal{H}(z)$. For $\theta^{0i} = 0$, it has been described in [1, 7] how to derive Feynman rules and the outcome showed, that the usual causal Feynman propagator can be used. However, the interaction vertices are modified by certain phase factors. The existence of such phase factors leads to planar and non-planar contributions. But furthermore, it was pointed out in [7] that the case $\theta^{0i} \neq 0$ is difficult to handle due to the fact that the lagrangian containing star products of fields consequently also depends on infinitely many time derivatives acting on fields. Thus, it may be doubted that the lagrangian formalism can be applied for $\theta^{0i} \neq 0$ in the usual, traditional way.

An alternative approach was already proposed in [11] and analysed explicitely in [8, 10]. The calculations carried out there are based on the Gell-Mann-Low formula

$$\begin{aligned}\langle 0|\, T\{\phi(x_1)\ldots\phi(x_n)\}\,|0\rangle_H = &\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\cdots\int_{-\infty}^{\infty}dt_m\,\times\\ &\times\,\langle 0|\, T\{\phi(x_1)\cdots\phi(x_n)V(t_1)\cdots V(t_m)\}\,|0\rangle_0\,.\end{aligned} \tag{2.4}$$

The subscript H indicates that the operators are given in the Heisenberg picture and the subscript 0 refers to the interaction picture. Following the derivation of this formula (see [16] for example), it is clear that time ordering is to be done with respect to $x_1^0, \ldots, x_n^0$ and $t_1, \ldots, t_m$, called *time stamps* from now on. In order to apply eq. (2.4) for the interaction given by eq. (2.3), it is helpful to rewrite $\mathcal{H}$ in the form of integral representations [10]

$$\begin{aligned}\mathcal{H}(z) = &\frac{\kappa}{4!}\int\prod_{i=1}^{3}\left(d^4s_i\frac{d^4l_i}{(2\pi)^4}e^{il_is_i}\right)\phi\left(z-\frac{1}{2}\tilde{l}_1\right)\phi\left(z+s_1-\frac{1}{2}\tilde{l}_2\right)\phi\left(z+s_1+s_2-\frac{1}{2}\tilde{l}_3\right)\times\\ &\times\phi\left(z+s_1+s_2+s_3\right),\end{aligned} \tag{2.5}$$

where $\tilde{a}^\nu \equiv a_\mu \theta^{\mu\nu}$. Time ordering only involves z^0 and not any other time components occurring in the field operators in eq. (2.5). The advantages of this representation are that the non-locality can be nicely seen and that one does not have to care about derivatives, especially time derivatives, any more as it is the case when explicitly using the star product in the form of eq. (2.2).

Now, the question is, how to do perturbation theory with interactions like in eq. (2.5)? The case of $\theta_{0i} \neq 0$ was already treated in [8, 10] and rules how to do calculations were also given. But as was also pointed out, combinatorics was not explicitly treated. Furthermore, it is hard to see the connection between the rules given in [8, 10] and ordinary Feynman rules, which should come out for the case $\theta = 0$. Therefore, we have developed graphical rules in the fashion of Feynman rules for perturbation theory of non-local interactions of scalar particles of the type

$$V(t) = \int d\underline{\lambda}\, v(\underline{\lambda}, t)\phi(g_1(\underline{\lambda}, t)) \cdots \phi(g_k(\underline{\lambda}, t)) \tag{2.6}$$

in the following section. $\underline{\lambda} = (\lambda_1, \ldots, \lambda_e)$ denotes a set of e real parameters, v is a C-function and the g_i's map $(\underline{\lambda}, t)$ into a four vector. $d\underline{\lambda}$ simply abbreviates $d\lambda_1 \ldots d\lambda_e$.

Concerning eq. (2.5), this means

$$\begin{aligned}
k &= 4\,,\\
\underline{\lambda} &= (\mathbf{z}, l_1, l_2, l_3, s_1, s_2, s_3)\,,\\
v(\underline{\lambda}, t) &= \frac{\kappa}{4!}\,(2\pi)^{-12}\, e^{i(l_1 s_1 + l_2 s_2 + l_3 s_3)}\,,\\
g_1(\underline{\lambda}, t) &= z - \frac{\tilde{l}_1}{2}\,,\\
g_2(\underline{\lambda}, t) &= z + s_1 - \frac{\tilde{l}_2}{2}\,,\\
g_3(\underline{\lambda}, t) &= z + s_1 + s_2 - \frac{\tilde{l}_3}{2}\,,\\
g_4(\underline{\lambda}, t) &= z + s_1 + s_2 + s_3\,,
\end{aligned}$$

with $t = z^0$. The non-local interaction studied in [15] was the main motivation for attacking the problem in such a general way:

$$\begin{aligned}
V_k^B(z^0) = \kappa\, c_k \int d^3 z \int d^4 a_1 \cdots d^4 a_k \ :\phi(z + \zeta a_1) \cdots \phi(z + \zeta a_k):\ \times\\
\times \exp\left\{-\frac{1}{2}\sum_{j=1}^{k} (a_j^\mu)^2\right\}\, \delta^4\Big(\frac{1}{k}\sum_{j=1}^{k} a_j\Big)\,,
\end{aligned} \tag{2.7}$$

with $\kappa, c_k, \zeta \in R$. κ is the coupling constant and c_k denotes some normalization constant depending on k. ζ has a physical dimension of a length and should be very small (maybe in the range of the Planck length). We have explicitly introduced this parameter, since the limit $\zeta \to 0$ represents the corresponding local theory. We will refer to interactions of this type as *non-local interactions of gaussian type* from now on. Besides the usual normal ordering indicated by the colons, this can be put into the form of eq. (2.6):

$$
\begin{aligned}
\underline{\lambda} &= (\mathbf{z}, a_1, \dots a_k)\,, \\
v(\underline{\lambda}, t) &= \kappa\, c_k \exp\left\{ -\frac{1}{2}\sum_{j=1}^{k} (a_j^\mu)^2 \right\} \delta^4\Big(\frac{1}{k}\sum_{j=1}^{k} a_j\Big)\,, \\
g_j(\underline{\lambda}, t) &= z + \zeta a_j\,.
\end{aligned}
$$

with $j = 1, \dots, k$ and $z^0 = t$. The effect of normal ordering will be discussed below.

Before applying the Gell-Mann-Low formula (2.4) to these interactions, it must be mentioned that $V(t)$ should have a time dependence according to the interaction picture:

$$V(t) = \exp(iH_0 t) V(0) \exp(-iH_0 t)\,. \tag{2.8}$$

Choosing ϕ in the interaction picture, this relation can be easily satisfied by

$$v(\underline{\lambda}, t) = v(\underline{\lambda}, 0)\,, \tag{2.9}$$

$$g_i(\underline{\lambda}, t)^T = (g_i(\underline{\lambda}, 0)^1, g_i(\underline{\lambda}, 0)^2, g_i(\underline{\lambda}, 0)^3, g_i(\underline{\lambda}, 0)^0 + t)\,. \tag{2.10}$$

The interactions of eqs. (2.5) and (2.7) obey these conditions. Note that we have adopted the convention

$$x^\mu = (x, y, z, t)^T = (x^1, x^2, x^3, x^0)^T$$

and we make use of the metric defined by

$$a_\mu = (a_1, a_2, a_3, a_0) \equiv (a^1, a^2, a^3, -a^0)\,.$$

3. Feynman rules for non-local interactions

In this section, diagrammatic rules will be given for calculating

$$G_m^n(x_1, \dots, x_n) \equiv \frac{(-i)^m}{m!} \int dt_{n+1} \cdots dt_N \, \langle 0|\, T\{\phi(x_1)\cdots\phi(x_n) V(t_{n+1}) \dots V(t_N)\}\, |0\rangle_0\,, \tag{3.1}$$

with non-local interactions as given in eq. (2.6). n is the number of external points and m denotes the order of interactions. $N = n + m$ and the time stamps $t_1, \dots, t_m$ have been renamed by $t_{n+1}, \dots, t_{n+m}$. In order to understand the general case which will be discussed in appendix A, we consider in a first step the simplest non-local interaction of the following form:

$$V(t) = \int d\underline{\lambda}\, v(\underline{\lambda}, t) \phi(g_1(\underline{\lambda}, t)) \phi(g_2(\underline{\lambda}, t))\,.$$

Such an interaction could be responsible for "mass-renormalization" if one investigates the connected piece of a "physical" two-point function

$$G_1^2(x, y) = \int dt \int d\underline{\lambda}\, v(\underline{\lambda}, t)\, \langle 0|\, T\{\phi(x)\phi(y)\phi(g_1(\underline{\lambda}, t))\phi(g_2(\underline{\lambda}, t))\}\, |0\rangle_0\,. \tag{3.2}$$

As pointed out in [8, 9, 10, 13], one emphasizes here that the time ordering involves only the times x^0, y^0 and the time stamp of the interaction part due to the unitarity problem which is discussed in [12, 13]. Thus, time ordering is not done with respect to $g_i(\underline{\lambda}, t)^0$, but simply t.

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Abbreviating $g_1(\underline{\lambda}, t) = z + a$ and $g_2(\underline{\lambda}, t) = z + b$, with t only occurring in $z^0 = t$, eq. (2.10) is automatically satisfied. We are now able to evaluate eq. (3.2) in terms of free field commutators of $\phi^+(x)$ and $\phi^-(x)$. In pushing annihilation operators $\phi^+(x)$ to the right or creation operators $\phi^-(y)$ to the left in using

$$\phi(x) = \phi^+(x) + \phi^-(x)\,, \tag{3.3}$$

$$\left[\phi^+(x), \phi^-(y)\right] \equiv \Delta^+(x, y)\,, \tag{3.4}$$

where we use the following conventions

$$\phi^+(x) = (2\pi)^{-3/2} \int d^3p \frac{e^{ip^+x}}{\sqrt{2\omega_{\mathbf{p}}}}\, a(\mathbf{p})\,, \tag{3.5}$$

$$\phi^-(x) = (2\pi)^{-3/2} \int d^3p \frac{e^{-ip^+x}}{\sqrt{2\omega_{\mathbf{p}}}}\, a^\dagger(\mathbf{p})\,, \tag{3.6}$$

$$\left[a^\dagger(\mathbf{p}), a(\mathbf{q})\right] = \delta^3(\mathbf{p} - \mathbf{q})\,, \tag{3.7}$$

where $\omega_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m^2}$ and $q^\sigma \equiv (\mathbf{q}, \sigma\omega_{\mathbf{q}})^T$, the connected part of the vacuum expectation value can now be written as

$$\begin{aligned}
&\langle 0|\, T\{\phi(x)\phi(y)\phi(z+a)\phi(z+b)\}\, |0\rangle_0^{\text{con}} = \qquad (3.8)\\
&= \theta(x^0 - y^0)\theta(y^0 - z^0)\left[\Delta^+(x, z+a)\Delta^+(y, z+b) + \Delta^+(x, z+b)\Delta^+(y, z+a)\right] +\\
&\quad + \theta(y^0 - x^0)\theta(x^0 - z^0)\left[\Delta^+(x, z+a)\Delta^+(y, z+b) + \Delta^+(x, z+b)\Delta^+(y, z+a)\right] +\\
&\quad + \theta(x^0 - z^0)\theta(z^0 - y^0)\left[\Delta^+(x, z+a)\Delta^+(z+b, y) + \Delta^+(x, z+b)\Delta^+(z+a, y)\right] +\\
&\quad + \theta(y^0 - z^0)\theta(z^0 - x^0)\left[\Delta^+(z+a, x)\Delta^+(y, z+b) + \Delta^+(z+b, x)\Delta^+(y, z+a)\right] +\\
&\quad + \theta(z^0 - x^0)\theta(x^0 - y^0)\left[\Delta^+(z+a, x)\Delta^+(z+b, y) + \Delta^+(z+b, x)\Delta^+(z+a, y)\right] +\\
&\quad + \theta(z^0 - y^0)\theta(y^0 - x^0)\left[\Delta^+(z+a, x)\Delta^+(z+b, y) + \Delta^+(z+b, x)\Delta^+(z+a, y)\right].
\end{aligned}$$

The first (second) summands appearing in the square brackets above will be referred to as the uncrossed (crossed) terms. The crossed terms can be obtained from the uncrossed ones by simply exchanging $x \leftrightarrow y$ or $a \leftrightarrow b$. Now, let us pick out the uncrossed terms containing $\Delta^+(y, z+b)$ of eq. (3.8):

$$\begin{aligned}
\Delta^+(y, z+b)[&\theta(x^0 - y^0)\theta(y^0 - z^0)\Delta^+(x, z+a) + \theta(y^0 - x^0)\theta(x^0 - z^0)\Delta^+(x, z+a) +\\
&+ \theta(y^0 - z^0)\theta(z^0 - x^0)\Delta^+(z+a, x)]\,.
\end{aligned}$$

For $(x^0 \neq y^0 \neq t)$,[1] one can now apply a trick: multiply the first term in square brackets with $\theta(x^0 - z^0)$, the second with $\theta(y^0 - z^0)$, the third with $\theta(y^0 - x^0)$, and add within the square brackets $\theta(y^0 - z^0)\theta(z^0 - x^0)\theta(x^0 - y^0)\Delta^+(z+a, x)$, which is 0. Thus, one does not alter the result and gets

$$\theta(y^0 - z^0)\Delta^+(y, z+b)\left\{\theta(x^0 - z^0)\Delta^+(x, z+a) + \theta(z^0 - x^0)\Delta^+(z+a, x)\right\}.$$

[1]The contributions where some time stamps are the same can be neglected.

The expression in curly brackets clearly reminds of the usual Feynman propagator

$$-i\Delta_F(x,x') = \theta(x^0 - x'^0)\Delta^+(x,x') + \theta(x'^0 - x^0)\Delta^+(x',x)\,. \tag{3.9}$$

For $a = 0$, which would be the case for a local field theory, it reduces to Δ_F. Therefore, we will define

$$-\,i\Delta(x,t;x',t') \equiv \theta(t-t')\Delta^+(x,x') + \theta(t'-t)\Delta^+(x',x) \tag{3.10}$$

and call Δ the *contractor*. The name is created in analogy to the usual Wick-contractions of commutative local perturbation theory in the sense of Feynman. The semicolon just visualizes the connection between four vectors x or x' and t and t', respectively. Treating the remaining terms of eq. (3.8) in the same way, G_1^2 can now be rewritten as

$$\begin{aligned} G_1^2(x,y) = -\int dt \int d\underline{\lambda}\, v(\underline{\lambda},t) \times [&\Delta(x,x^0;g_1(\underline{\lambda},t),t)\Delta(y,y^0;g_2(\underline{\lambda},t),t) + \\ &+ \Delta(x,x^0;g_2(\underline{\lambda},t),t)\Delta(y,y^0;g_1(\underline{\lambda},t),t)]\,. \end{aligned}$$

This example will be further discussed in section 4. Before, the diagrammatic rules for the general case should be given.

3.1 Coordinate space rules

Again, we refer to appendix A, where the general case including beside the connected parts also tadpole contributions and disconnected diagrams has been treated. In order to calculate $G_m^n(x_1,\dots,x_n)$, one has to apply the following diagrammatic algorithm:

- Draw n points and label them with the external coordinates $x_1,\dots,x_n$. Their time stamps are $x_1^0,\dots,x_n^0$, respectively.
- Draw m circles and label them with the parameters $\underline{\lambda}_1,\dots,\underline{\lambda}_m$ and time stamps $t_{n+1},\dots,t_{n+m}$.
- Draw k points into each circle and label them with $g_1,\dots,g_k$.
- For each possibility of connecting two points pairwise by a line, so that each point is connected to exactly one line, draw a diagram with points and circles as given above.
- For each line connecting two points with coordinates and time stamp x, t and x', t', respectively, write down a contractor

  $$-i\Delta(x,t;x',t')\,,$$

  if the points do not belong to the same circle. If they belong to the same circle, write down

  $$\text{either}\quad \Delta^+(x,x') \quad\text{or}\quad \Delta^+(x',x)\,,$$

  depending on whether $\phi(x)$ stands left of $\phi(x')$ within the interaction $V(t)$ or vice versa. External points already carry the coordinates as label and the corresponding time stamp is simply the 0th component of that label. For points within circles, the time stamp t is simply the time stamp of the circle. The coordinate x of such a point is given by the label $\underline{\lambda}$ and t of the circle and the label g_j of the point as $x = g_j(\underline{\lambda},t)$.

- For each circle labeled with $\underline{\lambda}_i$ and t_{n+i}, perform an integration according to

$$(-i)\int dt_{n+i}\, d\underline{\lambda}_i v(\underline{\lambda}_i, t_{n+i})\,.$$

- Sum up the contributions of all diagrams.

The rules given above are considerably more complicated than the usual Feynman rules of the corresponding local field theory. But using this diagrammatic formalism, eq. (A.13) is much more comfortable to handle than the usual algorithm of commuting out all creation and annihilation fields. It should also be mentioned that for each diagram with m identical vertices, there are $m!$ diagrams which only differ by a rearrangement of the vertices. This fact has already been implemented in the rules to cancel the factor $1/m!$ of the Gell-Mann-Low formula (2.4). Consequently, one must include diagrams differing by a rearrangement of vertices exactly once.

Further simplifications might be possible when permuting the labels g_i of a given circle among each other. But this will in general depend on the interaction itself and will be studied below for a more special type of interactions. Here it is a good point to comment on the prescription for calculating tadpole contributions. The rules given above mean that for a tadpole loop between points with coordinates x and x', one has to include either $\Delta^+(x,x')$ or $\Delta^+(x',x)$, depending on the definition of the interaction. If the time stamp of the circle is t, the contractor for these coordinates would give

$$-i\Delta(x,t;x',t) = \theta(0)\left[\Delta^+(x,x') + \Delta^+(x',x)\right]$$

which clearly is not what one needs for the general case. But in a local field theory, $\Delta^+(x,x)$ is to be taken for tadpoles loops. Thus, by defining $\theta(0) \equiv 1/2$, one does not have to treat tadpole contributions exceptionally, but can also use the contractor $-i\Delta(x,t;x,t) = \Delta^+(x,x)$ in local field theories.

3.2 Calculation of the contractor

In order to evaluate the contractor, one uses eqs. (3.5),(3.6) and (3.7) to get

$$\Delta^+(x,x') = \left[\phi^+(x), \phi^-(x')\right] = (2\pi)^{-3}\int \frac{d^3p}{2\omega_{\mathbf{p}}} e^{ip^+(x-x')}\,. \tag{3.11}$$

Expressing the θ-function as

$$\theta(t) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} ds \frac{e^{-ist}}{s+i\epsilon}\,, \tag{3.12}$$

one obtains

$$\begin{aligned}
\theta(t-t')\Delta^+(x,x') &= \frac{i}{(2\pi)^4}\int \frac{d^3p ds}{2\omega_{\mathbf{p}}} e^{ip^+(x-x')-is(x^0-x'^0)} \frac{e^{is(x^0-t-(x'^0-t'))}}{s+i\epsilon} \\
&= \frac{i}{(2\pi)^4}\int \frac{d^4p}{2\omega_{\mathbf{p}}} e^{ip(x-x')+ip^0(x^0-t-(x'^0-t'))} \frac{e^{-i\omega_{\mathbf{p}}(x^0-t-(x'^0-t'))}}{p^0-\omega_{\mathbf{p}}+i\epsilon}\,.
\end{aligned}$$

The last line was obtained by the transformation $s \equiv p^0 - \omega_{\mathbf{p}}$. The other term of eq. (3.10) can be obtained by simply exchanging (x, t) and (x', t'), and additionally one transforms $p \to -p$. The obtained expression only differs in the last fraction which is

$$\frac{e^{i\omega_{\mathbf{p}}(x^0 - t - (x'^0 - t'))}}{-p^0 - \omega_{\mathbf{p}} + i\epsilon} .$$

Thus, one finally gets

$$\begin{aligned} \Delta(x, t; x', t') &= \frac{1}{(2\pi)^4} \int d^4q \frac{e^{iq(x-x')+iq^0(x^0-t-(x'^0-t'))}}{q^2 + m^2 - i\epsilon} \times \\ &\times \left[\cos(\omega_{\mathbf{q}}(x^0 - t - (x'^0 - t'))) - \frac{iq^0}{\omega_{\mathbf{q}}} \sin(\omega_{\mathbf{q}}(x^0 - t - (x'^0 - t'))) \right] . \end{aligned}$$

For theories, where the time stamps are always identical with the time components of field arguments, say $t = x^0$ and $t' = x'^0$, this reduces to the usual Feynman propagator

$$\Delta_F(x, x') = \frac{1}{(2\pi)^4} \int d^4q \frac{e^{iq(x-x')}}{q^2 + m^2 - i\epsilon} .$$

3.3 Energy-momentum conservation

Now we present the calculation of diagrams as shown in figure 1, for example, using translationally invariant interactions. Figure 1 consists of one interaction vertex containing k points labeled g_i with $i = 1, \ldots, k$. These points all carry the same time stamp. The k "external" points are labeled $z_i + a_i$ all carrying different time stamps z_i^0 $(i = 1, \ldots, k)$. Actually, a typical external point would carry time stamp $z_i^0 + a_i^0$. But the intension of this calculation is to work out Feynman rules in more detail. The "external" points are thus kept more general in order to be also usable as internal points of larger diagrams containing figure 1. The very general interaction defined by eq. (2.6) is now specialized in order to be translationally invariant:

$$\underline{\lambda} = (\mathbf{z}, \underline{\mu}) , \tag{3.13}$$

$$v(\underline{\lambda}, t) = w(\underline{\mu}) , \tag{3.14}$$

$$g_i(\underline{\lambda}, t) = z + h_i(\underline{\mu}) , \tag{3.15}$$

where bold symbols denote three-vectors and $z^0 \equiv t$. $\underline{\mu} = (\mu_1, \ldots, \mu_{e-3})$ denotes a set of $e - 3$ real parameters, w is a C-function and the h_i's map $\underline{\mu}$ into a four vector. $d\underline{\mu}$ simply abbreviates $d\mu_1 \ldots d\mu_{e-3}$. The very general interaction defined by eq. (2.6) is thus specialized slightly:

$$V(z^0) = \int d^3z \int d\underline{\mu}\; w(\underline{\mu}) \phi(z + h_1(\underline{\mu})) \cdots \phi(z + h_k(\underline{\mu})) . \tag{3.16}$$

This way, translation invariance is manifested:

$$e^{-i\epsilon P_{(0)}} V\left(z^0\right)\; e^{i\epsilon P_{(0)}} = V\left(z^0 + \epsilon^0\right) , \tag{3.17}$$

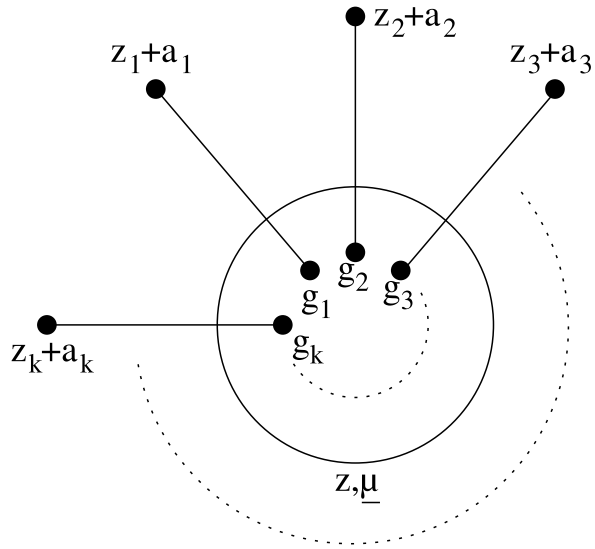

Figure 1: Vertex of an interaction with k fields. The points out of the circle are considered to be either external or part of other vertices. Their time stamps are $z_1^0, \ldots, z_k^0$. The time stamp of the circle is z_0.

where $P_{(0)}$ denotes the free four-momentum operator generating translations. ϵ represents the constant translation in coordinate space. This way, eq. (2.8) is automatically satisfied.

Concerning the non-commutative interaction given by eq. (2.5), the above specialization means

$$\begin{aligned}
k &= 4\,, \\
\underline{\mu} &= (l_1, l_2, l_3, s_1, s_2, s_3)\,, \\
w(\underline{\mu}) &= \frac{\kappa}{(4!(2\pi)^4)} \exp\left(i(l_1 s_1 + l_2 s_2 + l_3 s_3)\right), \\
h_1(\underline{\mu}, t) &= -\frac{\tilde{l}_1}{2}\,, \\
h_2(\underline{\mu}, t) &= s_1 - \frac{\tilde{l}_2}{2}\,, \\
h_3(\underline{\mu}, t) &= s_1 + s_2 - \frac{\tilde{l}_3}{2}\,, \\
h_4(\underline{\mu}, t) &= s_1 + s_2 + s_3\,, \\
\int d\underline{\mu} &= \int d^3 z \prod_{i=1}^{3} \int d^4 s_i \int d^4 l_i\,.
\end{aligned}$$

Now we are able to write down the contribution of the diagram shown in figure 1:

$$\begin{aligned}
G_I(z_1 + a_1, z_1^0; \ldots; z_k + a_k, z_k^0) \equiv -i \int d\underline{\mu}\, d^4 z\ w(\underline{\mu}) \left(-i\Delta(z_1 + a_1, z_1^0; z + h_1(\underline{\mu}), z^0\right) \cdots \\
\cdots \left(-i\Delta(z_k + a_k, z_k^0; z + h_k(\underline{\mu}), z^0\right)\,,
\end{aligned}$$

where I stands for the identity permutation of S^k, and S^k is the group of permutations of the integers $\{1,\dots,k\}$ (see also [17] for more details on permutations). G_I is only one certain contribution. On the whole, there are $k!$ terms similar to G_I but only differing in the way the external points are connected to the internal ones. Denoting an arbitrary permutation by $Q \in S^k$, one can write these terms as

$$G_Q(z_1+a_1,z_1^0;\dots;z_k+a_k,z_k^0) \equiv -i\int d\underline{\mu}\, d^4z\; w(\underline{\mu})\left(-i\Delta(z_1+a_1,z_1^0;z+h_{Q_1}(\underline{\mu}),z^0\right)\cdots$$
$$\cdots\left(-i\Delta(z_k+a_k,z_k^0;z+h_{Q_k}(\underline{\mu}),z^0\right).$$

G_Q gives the vertex where each point z_i+a_i is attached to $g_{Q_i}(\underline{\lambda},t) = z+h_{Q_i}(\underline{\mu})$. The total contribution of these terms is given by

$$G(z_1+a_1,z_1^0;\dots;z_k+a_k,z_k^0) \equiv \sum_{Q\in S^k} G_Q(z_1+a_1,z_1^0;\dots;z_k+a_k,z_k^0)\,. \tag{3.18}$$

All contractors connecting external and internal points are of the following form:

$$\Delta(z+a,z^0;z'+a',z'^0) = \frac{1}{(2\pi)^4}\int d^4q\frac{e^{iq(z-z')}}{q^2+m^2-i\epsilon}\sum_{\sigma\in\{1,-1\}}\frac{\omega_{\mathbf{q}}+\sigma q^0}{2\omega_{\mathbf{q}}}e^{iq^\sigma(a-a')}\,. \tag{3.19}$$

It is remarkable that z and z' only occur in the first exponential. This expresses translation invariance for time ordered perturbation theory of non-local interactions also involving time. Integrating over z immediately yields energy-momentum conservation and one gets

$$\begin{aligned}
&G(z_1+a_1,z_1^0;\dots;z_k+a_k,z_k^0) = \\
&= \frac{(-i)^{k+1}}{(2\pi)^{4k}}\sum_{Q\in S^k}\int d\underline{\mu}\; w(\underline{\mu})\int d^4q_1\cdots d^4q_k\;(2\pi)^4\delta^4(q_1+\cdots+q_k)\times\\
&\times\prod_{i=1}^k\left(\frac{e^{iq_iz_i}}{q_i^2+m^2-i\epsilon}\sum_{\sigma_i}\frac{\omega_{\mathbf{q}_i}+\sigma_iq_i^0}{2\omega_{\mathbf{q}_i}}e^{iq_i^{\sigma_i}(a_i-h_{Q_i})}\right)\\
&= \frac{(-i)^{k+1}}{(2\pi)^{4k}}\int d^4q_1\cdots d^4q_k\;(2\pi)^4\delta^4(q_1+\cdots+q_k)\prod_{i=1}^k\left(\frac{e^{iq_iz_i}}{q_i^2+m^2-i\epsilon}\right)\times\\
&\times\sum_{\sigma_1,\dots,\sigma_k}\prod_{i=1}^k\left(\frac{\omega_{\mathbf{q}_i}+\sigma_iq_i^0}{2\omega_{\mathbf{q}_i}}e^{iq_i^{\sigma_i}a_i}\right)\chi(q_1^{\sigma_1},\dots,q_k^{\sigma_k})\,,
\end{aligned}$$

with

$$\chi(q_1^{\sigma_1},\dots,q_k^{\sigma_k}) \equiv \int d\underline{\mu}\; w(\underline{\mu})\sum_{Q\in S^k}\exp\left(-i\sum_{i=1}^k q_{Q_i}^{\sigma_{Q_i}}h_i(\underline{\mu})\right). \tag{3.20}$$

Of course, the calculated vertex is much more complicated than what one knows from local field theories. But the positive aspect is given by the fact that the details of non-locality only occur in the factor χ. For the case of NCQFT, it contains the phase factors leading to planar and non-planar contributions. χ can be said to be the central quantity of time ordered perturbation theory for non-local interactions obeying translation invariance.

Corresponding to this result, it is now straight forward to extract the momentum space Feynman rules for non-local interactions.

3.4 Momentum space rules

Usually, one is interested in the Fourier transform of eq. (2.4). Correspondingly, one has to evaluate[2]

$$
\begin{aligned}
G^{\mathrm{con,nt}}(p_1,\ldots,p_n) &\equiv \prod_{i=1}^{n}\left(\int d^4x_i\, e^{ip_ix_i}\right)\langle 0|\, T\{\phi(x_1)\ldots\phi(x_n)\}\,|0\rangle_H^{\mathrm{con,nt}} \\
&= (2\pi)^4\delta^4\left(\sum_{i=1}^{n} p_i\right)\prod_{i=1}^{n-1}\left(\int d^4z_i\, e^{ip_iz_i}\right)\times \\
&\quad\times \langle 0|\, T\{\phi(z_1)\ldots\phi(z_{n-1})\phi(0)\}\,|0\rangle_H^{\mathrm{con,nt}} \\
&\equiv (2\pi)^4\delta^4\left(\sum_{i=1}^{n} p_i\right)\, G^{\mathrm{con,nt}}_{\mathrm{tranc}}(p_1,\ldots,p_{n-1})\,.
\end{aligned}
\tag{3.21}
$$

This task can be simplified using the rules of the last section. The δ-function in eq. (3.21) represents the total momentum conservation under the assumption that all momenta are flowing into the graph. The main result of the calculation of the last section is the fact that summing up diagrams, which only differ by the way how the contractors are attached to points within fixed circles, can easily be done by including the factor χ of eq. (3.20) for each vertex. But, using χ, one has to take care of two things:

- When calculating $\sum_Q G_Q$, it was assumed that all momenta are flowing into the vertex. Thus, one has to insert $-(p^\sigma)$ into χ if p is flowing out of the vertex.
- Furthermore, for certain diagrams, one would get double counting. Thus one has to include a symmetry factor $1/S$ (see also [8]). It must be taken the same as for common Feynman diagrams.

Before presenting the final rules, we will briefly repeat topological considerations. A general graph is characterized by a certain number of internal lines I and by the number V of interaction vertices of a given type. The number of independent loops are denoted by L, which is

$$L = I - V + 1\,. \tag{3.22}$$

With the above momentum assignment that all momenta are incoming, we are now able to state the Feynman rules for the calculation of expression (3.21):

1. Draw all possible momentum space Feynman diagrams having $E = n$ external lines.
2. Carefully label each line with four momentum including its flow and make use of the conservation of four momentum at each vertex. Due to definition (3.21), the external lines are labeled with momenta $p_1,\ldots,p_n$ with the convention that the p_i's are incoming. Also assign a variable σ_i to each line.

[2]The superscript con, nt indicates that we restrict ourselves to connected diagrams without tadpoles. The discussion of tadpoles will be treated as an example in section 4.

3. For each line, one has to include a factor

$$\frac{-i}{q^2+m^2-i\epsilon}\,\frac{\omega_{\mathbf{q}}+\sigma q^0}{2\omega_{\mathbf{q}}}\,,$$

where q and σ now represent the labels of the corresponding line.

4. For each vertex, write down a factor

$$-i\,\chi(\ldots)$$

with the rule to insert $\pm(q_i^{\sigma_i}) = (\pm\mathbf{q}_i, \pm\sigma_i\omega_{\mathbf{q}_i})^T$ into χ for each line (at the vertex) labeled q_i, σ_i; the "+" sign for momenta flowing into the vertex and "−" otherwise. Due to the symmetry of χ concerning permutations of arguments, the order of arguments is not relevant.

5. Include the symmetry factor

$$\frac{1}{S}\,.$$

6. Assure momentum conservation by a factor

$$(2\pi)^4\delta^4(p_1+\cdots+p_n)\,.$$

7. Integrate over the L independent loop momenta, which are not fixed by energy-momentum conservation and multiply by $(2\pi)^{-4L}$. Sum over all σ's.

8. Sum up all diagrams in the usual sense.

Now one is ready to apply the result of the rather general treatment of time ordered perturbation theory valid for a large class of non-local interactions to special cases.

4. Applications

4.1 Non-commutative interactions

In this section, we study interactions of the following type:

$$V_k(z^0) = \frac{\kappa}{k!}\int d^3z\;(\phi(z))^{*k}\,, \tag{4.1}$$

where $*$ in the power indicates that the star product is to be used between the k fields. For $k=4$, this is just the interaction density given in eq. (2.3) integrated over d^3z. In general, one can write

$$V_k(z^0) = \frac{\kappa}{k!}\int d^3z\,\prod_{i=1}^{k-1}\left(\int\frac{d^4l_i\,d^4s_i}{(2\pi)^4}\,e^{il_is_i}\phi\left(z-\frac{1}{2}\tilde{l}_i+\sum_{j=1}^{i-1}s_j\right)\right)\phi\left(z+\sum_{j=1}^{k-1}s_j\right)\,. \tag{4.2}$$

These interactions are of the type as specified in eqs. (3.13-3.15) and thus obey momentum conservation. The factor χ of eq. (3.20) can then be evaluated for the interactions V_k (abbreviating $p_i \equiv q_i^{\sigma_i}$):

$$\chi_k(p_1, \ldots, p_k) = \frac{\kappa}{k!} \sum_{Q \in S^k} \exp\left(-i \sum_{i<j} p_{Q_i} \wedge p_{Q_j}\right) \tag{4.3}$$

with

$$a \wedge b \equiv \frac{1}{2}\theta_{\mu\nu}\, a^\mu b^\nu = -\frac{1}{2}\, a\, \tilde{b}\,.$$

These results can be compared to [8], and they agree.

The result for $k = 2$ is remarkable. It can be written as

$$G_1^2(p,q)^{\rm con} = \kappa\delta(p+q)\frac{(-i)^3(2\pi)^4}{(p^2+m^2-i\epsilon)^2}\frac{\omega_{\mathbf{p}}{}^2 + p^{0\,2} + (\omega_{\mathbf{p}}{}^2 - p^{0\,2})\cos(p^+ \wedge p^-)}{2\omega_{\mathbf{p}}{}^2}\,. \tag{4.4}$$

It is surprising that this term contains a phase factor which only vanishes for $\theta_{0i} = 0$. On-shell, it does not have any unusual effect since $\omega_{\mathbf{p}}{}^2 = (p^0)^2$. But off-shell it might be used as an extra counterterm within loops or other internal lines. If one starts with the naive lagrangian approach, quadratic terms do not contain phases since

$$\int d^4x\,(f * g)(x) = \int d^4x\; f(x)\, g(x)\,.$$

This means that the free action is the same as ordinarily and the free hamiltonian of non-commutative theory is the same as in the commutative case. Apparently, this might seem problematic for our work since we used H_0 of the commutative case as free hamiltonian. But this problem can be resolved the following way: Suppose, there was some non-commutative full hamiltonian H_*. Then one can simply extract the interaction V by definition:

$$V \equiv H_* - H_0\,.$$

The question of In- and Out-states and their asymptotic behaviour, which is important for the physical meaning of the presented calculations, must be discussed elsewhere. The result presented in eq. (4.4) shows that the naive lagrangian approach and time ordered perturbation theory for non-local interactions worked out here are not equivalent when non-locality also involves time ($\theta_{0i} \neq 0$). For the case $\theta_{0i} = 0$, it is easy to verify the agreement of the rules given here and the usual treatment (see [7] for example) by simply realizing that $a \wedge b$ is independent of a^0 and b^0.

Finally, the calculation of a tadpole according to the coordinate space rules should be given, namely

$$G_{\rm tp}(x,y) \equiv -i \int dz^0\;\; \langle 0|\, T\{\phi(x)\phi(y)V_4(z^0)\}\,|0\rangle_H^{\rm con}\;.$$

In this case, all connected contributions contain exactly one line starting and ending in the circle representing the interaction, which means that one only has diagrams containing tadpoles. Thus we cannot use the momentum space rules which have not been

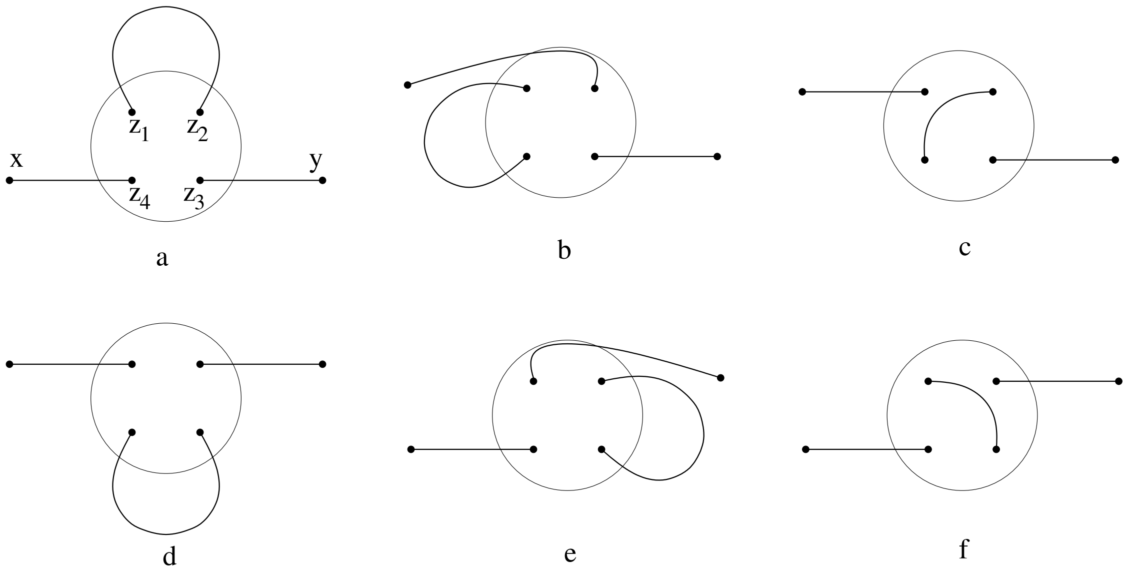

Figure 2: This figure shows half of the diagrams contributing to $G_{\mathrm{tp}}(x,y)$. The crossed diagrams which are obtained by exchanging x and y have been omitted. Diagrams b - f are considered to be labeled like a. We have used the abbreviations $z_1 = z - \frac{1}{2}\tilde{l}_1$, $z_2 = z - \frac{1}{2}\tilde{l}_2 + s_1$, $z_3 = z - \frac{1}{2}\tilde{l}_3 + s_1 + s_2$, $z_4 = z + s_1 + s_2 + s_3$.

worked out for tadpole contributions, but we have to take the somewhat lengthier coordinate space rules. We simply consider this as an exercise and check of our approach: using normal ordered interactions, there would not be tadpole contributions. Therefore, we do not see any directly physical relevance of such contributions. The result is obtained by calculating the diagrams shown in figure 2 and the corresponding crossed diagrams which are obtained by simply exchanging x and y. Diagram c and its crossed one give

$$-i\int d^4z \prod_{i=1}^{3}\left(\frac{d^4l_i d^4s_i}{(2\pi)^4}e^{il_is_i}\right)\left[\Delta^+(z_2,z_4)(-i\Delta(x,x^0;z_1,z^0))(-i\Delta(y,y^0;z_3,z^0)) + x\leftrightarrow y\right],$$

with $z_1,\ldots,z_4$ as described in figure 2. Summing up all 12 diagrams and carrying out the trivial integrations, one arrives at

$$\begin{aligned} G_{\mathrm{tp}}(x,y) = {} & \frac{i\kappa}{4!}\int \frac{d^4q}{(2\pi)^4{\omega_{\mathbf{q}}}^2}\frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}}\; e^{iq(x-y)} \times \\ & \times\left[\frac{2+\cos(q^+\tilde{k}^+)}{(q^0-\omega_{\mathbf{q}}+i\epsilon)^2} + \frac{2+\cos(q^-\tilde{k}^+)}{(q^0+\omega_{\mathbf{q}}-i\epsilon)^2} - \right. \\ & \left. - \cos\left(\frac{1}{2}q^-\tilde{q}^+\right)\frac{3+\cos(q^+\tilde{k}^+)+\cos((q^+-q^-)\tilde{k}^+)+\cos(q^-\tilde{k}^+)}{({q^0}^2-{\omega_{\mathbf{q}}}^2+i\epsilon)}\right]. \end{aligned} \tag{4.5}$$

This is consistent with the result of [10]. Diagrams c and f (and their crossed counterparts) shown in figure 2 are responsible for the contributions containing phase factors at the pole of order two at $q^0 \to \omega_{\mathbf{q}}$ of the integrand of eq. (4.5). Furthermore, it is interesting

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that the momentum q of external lines always occurs as $q^{\pm}$ in the cos-functions. For $\theta_{0i} = 0$, it would not make any difference whether we write $q^{\pm}\tilde{k^{\pm}}$ or $q\tilde{k}$, but for $\theta_{0i} \neq 0$, it definitely does. This new aspect also effects the UV/IR-mixing problem discussed in [14].

4.2 Non-local interactions of gaussian type

It can easily be seen that the interaction of eq. (2.7) proposed in [15] is translationally invariant in the sense of eqs. (3.13)–(3.15). So the momentum space rules are applicable. In order to specify perturbation theory further, one simply has to calculate χ of eq. (3.20):

$$\chi(p_1, \ldots, p_k) = \kappa c_k \int d^4a_1 \cdots d^4a_k \times \\ \times \exp\left\{-\frac{1}{2}\sum_{j=1}^{k}(a_j^{\mu})^2\right\}\delta^4\left(\frac{1}{k}\sum_{j=1}^{k}a_j\right)\sum_{Q\in S^k}\exp\left(-i\sum_{j=1}^{k}p_{Q_j}\,\zeta a_j\right), \quad (4.6)$$

where we have abbreviated $p_i \equiv q_i^{\sigma_i}$. A lengthy but standard calculation involving a multiple gaussian integral then yields (cf. [15])

$$\chi(p_1, \ldots, p_k) = \kappa\ (2\pi)^{2k-2}(k-1)!\, c_k\ e^{-k\,\zeta^2\left(\overline{(p^{\mu})^2}-\left(\overline{p^{\mu}}\right)^2\right)}, \quad (4.7)$$

with $\overline{A}$ representing the mean value

$$\overline{A} \equiv \frac{1}{k}\sum_{j=1}^{k}A_j\,.$$

We have written χ in such a statistical manner in order to see that the exponent in eq. (4.7) is always negative. Thus, it seems plausible that the exponential damping caused by χ makes the contribution of all diagrams finite. Indeed, $\int dx\, p_n(x)\ \exp(-x^2)$ is always finite for p_n a polynomial of order n. But one has to keep in mind that χ never involves time components q_i^0 of the four momenta q_i, but $q_i^{\sigma_i}$ instead. So it is clear that one separately has to check whether the integrations over time components are finite or not. Concerning the integrations over the 3-momenta, it seems plausible that they are finite due to the exponential damping. Investigations concerning these aspects are still to be carried out within the presented formalism, especially also for theories involving other types of particles.

It was shown in [15] that if the interaction is adiabatically switched off, all expectation values are finite at any order of perturbative expansion due to the presence of the gaussian kernels. In the proof, Wick's theorem was applied without the detour of giving explicit graphical rules.

5. Conclusion

The special case of NCQFT for $\theta_{0i} = 0$ can be generalized for other non-local interactions obeying translation invariance. Inspecting eq. (3.20), χ is independent of the time components of its arguments, if $h_i(\underline{\mu})^0 = 0$ for $1 \leq i \leq k$.

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Or in other words: *If time is not involved into non-locality, non-local interactions with translation invariance can be described by the usual Feynman rules with three-momentum-dependent factors to be included for each vertex.*

If time is involved into non-locality of a translationally invariant interaction, one has to use the momentum space rules given in 3.4. Such rules have in principal already been stated in [8] or [10]. The main difference between these rules and the ones presented here lies in the fact that we rearranged the θ-functions arising for time ordering T in a tricky way before blowing them up using eq. (3.12). Thus, we do not have to draw separate Feynman diagrams for each possible time ordering. This makes our rules much simpler to use. Our approach agrees with [8] who have calculated their results without making use of the non-local representation of the interaction. Furthermore we arrive at the same diagrammatic rules as [7] for $\theta_{0i} = 0$. Our results have been derived not only for NCQFT but for general non-local interactions of the type ϕ^k. The interaction $V(t)$ simply has to be invariant under time translations in the sense of eq. (2.8). If translation invariance is not satisfied for the spatial components, one is not lost because we can refer to the coordinate space rules presented in 3.1. An example for an interaction without translation invariance concerning spatial components would be the interaction

$$V(z^0) = \int d^3z\, U(\mathbf{z})\ \phi(z)^2\,.$$

Here, U is an unquantized, external potential constant in time scattering scalar particles. Except this simple example, one can assume that in general, these cases will be more complicated to handle.

Finally, it must be said that the naive lagrangian approach cannot be used when non-locality involves time. This can be seen in the case of NCQFT: If $\theta_{0i} \neq 0$, the lagrangian contains infinitely many time derivatives. How can canonical quantization work for these cases? The failure of the naive lagrangian approach is not a catastrophe since an alternative approach has already been followed in [8, 10] and also in our work. But the problem now is that this alternative approach only offers the possibility of studying symmetries perturbatively. When time is not involved into non-locality, the lagrangian formalism works in the case of NCQFT and maybe this is also true for other non-local interactions.

A. Generalized Wick-theorem for non-locally time ordered vacuum expectation values

In this section, the time ordered product of eq. (3.1)

$$\langle 0|\, T\{\phi(x_1)\cdots\phi(x_n)V(t_{n+1})\cdots V(t_N)\}\, |0\rangle_0 \tag{A.1}$$

will be further processed for non-local interactions as given in eq. (2.6). To deal with the expression given above, it is quite useful to define the times $t_1,\ldots,t_n$ as $x_1^0,\ldots,x_n^0$ and

$$\theta_P(\underline{t}) \equiv \theta(t_{P_1} - t_{P_2})\theta(t_{P_2} - t_{P_3})\cdots\theta(t_{P_{N-1}} - t_{P_N})\,, \tag{A.2}$$

where $\underline{t} \equiv (t_1, \dots, t_N)$ and $P \in S^N$, the group of permutations of $(1, \dots, N)$. P_i denotes the integer $P(i)$, the integer i is mapped to by the permutation P (see also [17] for more details on permutations). Inserting eq. (2.6), expression (A.1) can then be rewritten as

$$\int d\underline{\lambda}_1 \cdots d\underline{\lambda}_m v(\underline{\lambda}_1) \cdots \sum_{P \in S^N} \theta_P(\underline{t}) \langle 0| \phi(z_{\xi^P{}_1}) \cdots \phi(z_{\xi^P{}_M}) |0\rangle_0 \, , \tag{A.3}$$

where M is the number of fields occurring in the vacuum expectation value and $z_1, \dots, z_M$ are defined as the arguments put into the fields from left to right for the time ordering θ_P corresponding to the identity permutation $P = I$ (which corresponds to the time ordering $t_1 > t_2 > \cdots > t_N$). $z_{\xi^P{}_i}$ denotes the four vector which is put into the ith field according to the time ordering θ_P. Thus, ξ^P is implicitly defined as a permutation $\in S^M$ for each $P \in S^N$, so that $z_{\xi^P{}_i}$ is the argument put into the ith field (counting from left to right).

The vacuum expectation value in the above eq.

$$\langle 0| \phi(z_{\xi^P{}_1}) \cdots \phi(z_{\xi^P{}_M}) |0\rangle_0 \tag{A.4}$$

can be rewritten in terms of commutators by applying the following algorithm:

- Substitute every field operator $\phi(y)$ by the sum of its annihilation and creation part $\phi^+(y) + \phi^-(y)$.
- Expand the product into sums of products of ϕ^+ and ϕ^-.
- Expand each product by replacing adjacent pairs of operators $\phi^+(z_{\xi^P{}_i})\phi^-(z_{\xi^P{}_j})$ by $\phi^-(z_{\xi^P{}_j})\phi^+(z_{\xi^P{}_i}) + D_P(\xi^P{}_i, \xi^P{}_j)$, where[3]

  $$D_P(\xi^P{}_i, \xi^P{}_j) \equiv \theta(j - i) \left[\phi^+(z_{\xi^P{}_i}), \phi^-(z_{\xi^P{}_j})\right] \, . \tag{A.5}$$

  This step has to be repeated till no ϕ^- can be found right of any ϕ^+.
- Note that $\dots \phi^+(x) |0\rangle = \langle 0| \phi^-(x) \dots = 0$.

Now, the question is, which kind of terms remains after this procedure. First of all, it is clear that for M odd, the vacuum expectation value vanishes since for each summand only a single field operator would remain. So M even is the more interesting case. Some interesting properties for this case are:

- Each summand is a product of $M/2$ functions $D_P(\xi^P{}_i, \xi^P{}_j)$ with $i < j$ which each integer $\in \{1, \dots, M\}$ occurs exactly once in.
- Further on, all summands are different from each other, when considering $D_P(i, j)$ and $D_P(k, l)$ only identical for $i = k$ and $j = l$.

[3] The omission of $\theta(j - i)$ would not make any difference at this point since for all adjacent operators occurring during application of the given algorithm, $i < j$. Below, the purpose of this definition will become clear.

- Each product of $M/2$ functions $D_P({\xi^P}_i, {\xi^P}_j)$, which each integer $\in \{1, \dots, M\}$ occurs exactly once in, also represents a summand. The property $i < j$ of the first item would also imply a restriction on the integers i, j occurring in the arguments of D_P. But due to the apparently useless θ-function in eq. (A.5), terms not obeying this restriction are 0 which makes the restriction obsolete.

On the one hand, the first property says that one can write each summand S_i as

$$S_i = D_P({\xi^P}_{Q_1}, {\xi^P}_{Q_2}) D_P({\xi^P}_{Q_3}, {\xi^P}_{Q_4}) \cdots D_P({\xi^P}_{Q_{M-1}}, {\xi^P}_{Q_M}), \tag{A.6}$$

where $Q \in S^M$. But since ${\xi^P}_{Q_i} = \xi^P(Q(i))$ and $\xi^P \in S^M$, one can simply write $Q'_i = Q'(i) = \xi^P(Q(i))$ instead of ${\xi^P}_{Q_i}$. On the other hand, according to the third property, it can be said that each term $D_P(Q_1, Q_2) \dots D_P(Q_{M-1}, Q_M)$ with arbitrary Q occurs in the sum. The vacuum expectation value of eq. (A.4) can be written as

$$\langle 0| \phi(z_{{\xi^P}_1}) \cdots \phi(z_{{\xi^P}_M}) |0\rangle_0 = \frac{1}{(M/2)!} \sum_{Q \in S^M} D_P(Q_1, Q_2) \cdots D_P(Q_{M-1}, Q_M). \tag{A.7}$$

One has to divide by $(M/2)!$ since each product of $M/2$ functions with 2 arguments is generated by $(M/2)!$ permutations. Note that definition (A.5) is equivalent to

$$D_P(i,j) = \theta({(\xi^P)^{-1}}_j - {(\xi^P)^{-1}}_i) \left[\phi^+(z_i), \phi^-(z_j)\right]. \tag{A.8}$$

We have verified eq. (A.7) by carrying out a proof by induction. But due to its length and the fact that one can understand the result as described above, we want to skip it. Combining eq. (A.7) with the sum over permutations P due to time ordering, one gets

$$\begin{aligned}
&\sum_{P \in S^N} \theta_P(\underline{t}) \sum_{Q \in S^M} D_P(Q_1, Q_2) \dots D_P(Q_{M-1}, Q_M) = \\
&\quad = \sum_{Q \in S^M} C(Q_1, Q_2) \dots C(Q_{M-1}, Q_M) \;\times \\
&\quad\quad \times \sum_{P \in S^N} \theta_P(\underline{t})\; \theta\left({(\xi^P)^{-1}}_{Q_2} - {(\xi^P)^{-1}}_{Q_1}\right) \cdots \theta\left({(\xi^P)^{-1}}_{Q_M} - {(\xi^P)^{-1}}_{Q_{M-1}}\right),
\end{aligned} \tag{A.9}$$

where

$$C(i,j) \equiv \Delta^+(z_i, z_j). \tag{A.10}$$

The (only) nice thing about expression (A.9) is that the sum over P only depends on θ-functions. For given times $t_1, \dots, t_N$, θ_P now is only survived by one permutation . The other θ-functions oppose $M/2$ conditions

$${(\xi^P)^{-1}}_{Q_{2i-1}} < {(\xi^P)^{-1}}_{Q_{2i}},$$

or in other words: In expression (A.4), the field evaluated at $z_{Q_{2i-1}}$ had to stand left of the field evaluated at $z_{Q_{2i}}$. This condition can easily be rewritten in terms of the time stamps associated with the corresponding field arguments, provided the two stamps are not the same. Thus, it is comfortable to define τ_i as the mapping

$$\tau : \{1, \dots, M\} \to \{t_1, \dots, t_N\},$$

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where τ_i is defined to be the time stamp associated with the field argument z_i. For $\tau_{Q_{2i-1}} \neq \tau_{Q_{2i}}$ [4]

$$\theta\left(\left(\xi^P\right)^{-1}{}_{Q_{2i}} - \left(\xi^P\right)^{-1}{}_{Q_{2i-1}}\right) = \theta(\tau_{Q_{2i-1}} - \tau_{Q_{2i}})\,.$$

$\tau_{Q_{2i-1}} = \tau_{Q_{2i}}$ means that the two time stamps are the same and come from one interaction, say $D_P(Q_{2i-1}, Q_{2i})$ represents a tadpole loop. In this case, the θ-function simply assures that $\Delta^+(z_{Q_{2i-1}}, z_{Q_{2i}})$ is only taken into account when $\phi(z_{Q_{2i-1}})$ really stands left of $\phi(z_{Q_{2i}})$ in the interaction. It is useful to distinguish between sets of summands S^M_{nt} (no tadpole) where the above simplification is applicable and where it is not, S^M_t (tadpole):

$$S^M_{\mathrm{nt}} \equiv \left\{ Q \in S^M \middle| \forall i \in \left\{1, \ldots, \frac{M}{2}\right\} : \; \tau_{Q_{2i-1}} \neq \tau_{Q_{2i}} \right\}, \tag{A.11}$$

$$S^M_{\mathrm{t}} \equiv S^M \setminus S^M_{\mathrm{nt}}\,. \tag{A.12}$$

Summarizing the book keeping carried out above, one gets [5]

$$\begin{aligned}
&\langle 0|\, T\{\phi(x_1)\ldots\phi(x_n)V(t_{n+1})\ldots V(t_N)\}\, |0\rangle_0 = \qquad\qquad (A.13)\\
&= \int d\underline{\lambda}_1 \cdots d\underline{\lambda}_m v(\underline{\lambda}_1)\cdots \left[\frac{1}{(M/2)!2^{M/2}} \sum_{Q\in S^M_{\mathrm{nt}}} (-i\Delta(Q_1,Q_2))\cdots(-i\Delta(Q_{M-1},Q_M)) + \right.\\
&\qquad \left. + \frac{1}{(M/2)!} \sum_{P\in S^N} \theta_P(\underline{t}) \sum_{Q\in S^M_t} D_P(Q_1,Q_2)\cdots D_P(Q_{M-1},Q_M)\right]
\end{aligned}$$

where $\Delta(i,j)$ is the book keeping version of the contractor for integer arguments

$$-i\Delta(i,j) \equiv D_P(i,j) + D_P(j,i) = -i\Delta(z_i,\tau_i; z_j,\tau_j)\,.$$

The second sum still looks complicated, but one can use the contractor Δ for all lines but the tadpole lines. For the latter one, one simply has to insert Δ^+ with arguments corresponding to the order in the interaction. Eq. (A.13) now represents the generalized Wick theorem. It is based on eq. (A.7), which is in principle applicable for an arbitrary ordering of fields specified by the left hand side. Combining this relation with the time ordering supposed by the Gell-Mann-Low formula (2.4), one arrives at eq. (A.13), which we refer to the *generalized Wick theorem*. It is valid for non-local interactions, in particular, non-locality may also involve time. This means that $\phi(z_i)$ is not necessarily ordered with respect to z^0_i but τ_i. The ordinary Wick theorem is represented by the special case $\tau_i = z^0_i$. The contractor then becomes the usual propagator. For this case, our result agrees with the explicit Wick theorem (see [18]). As in this special case, the evaluation of eq. (A.13) for the general case can be simplified a lot by introducing diagrammatic rules. But in principle, it should be clear how this works and we confine ourselves to presenting the rules in section 3.

[4] This is meant to hold only for $\forall i,j \in \{1,\ldots,N\} : t_i \neq t_j$. The other cases should be irrelevant: Those, which involve time stamps associated with interactions have 0 Lebesgue measure, and those, where the time stamps of two external lines are the same, can be omitted anyway.

[5] In the first sum, the factor $1/(2^{M/2})$ arises from using the symmetric Δ which corresponds to blowing up the sum over Q by replacing $D_P(i,j)$ with $D_P(i,j) + D_P(j,i)$.

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